CS4311
Design and Analysis of Algorithms

Tutorial: KMP Algorithm

About this tutorial

- Introduce String Matching problem
- Knuth-Morris-Pratt (KMP) algorithm

String Matching

- Let T[0..n-1] be a text of length n
- Let P[0..p-1] be a pattern of length p
- Can we find all locations in T that P occurs?

- E.g., T = bacbababababacbb

  P = ababa

Here, P occurs at positions 4 and 6 in T

Brute Force Approach

- The easiest way to find the locations where P occurs in T is as follows:

  For each position of T

      Check if P occurs at that position

- Running time: worst-case O(np)

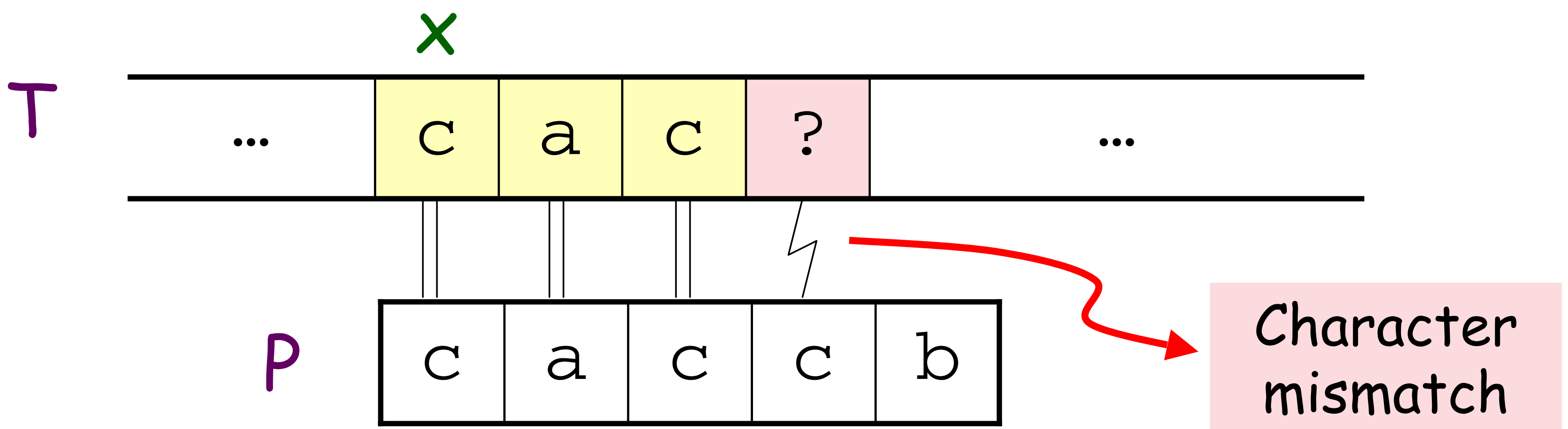
Brute Force Approach

- In the simple algorithm, when we decide that P does not occur at a position x, we start over to match P at position x+1

- However, even if P does not occur at position x, we may learn some information from this unsuccessful match
  ➔ may help to speed up later checking

Brute Force Approach

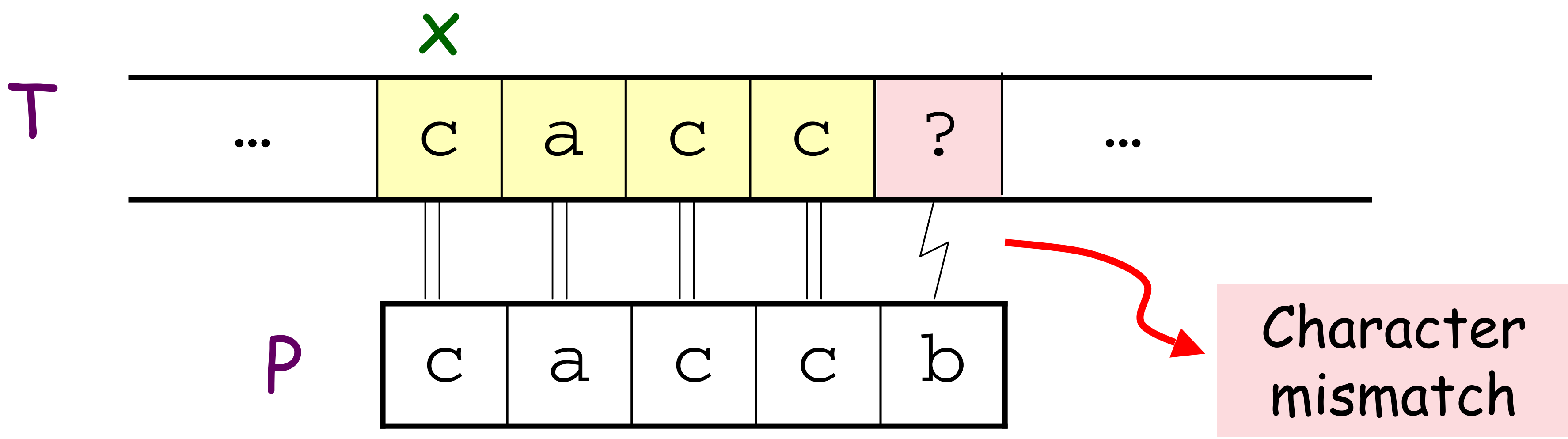
E.g., suppose when we check if P occurs at position x, we get the following scenario:



Can P occur in position x + 1 ?

Brute Force Approach

How about this case?



Can P occur in positions x+1, x+2, or x+3?

Key Observation

Lemma:

Suppose P has matched k chars with T[x...],
but has a mismatch at the $(k+1)^{th}$ char

That is, P[0..k-1] = T[x..x+k-1],

but P[k] $\neq$ T[x+k]

Then, for any $0 < r < k$,

if T[x+r...x+k-1] is not a prefix of P,

P cannot occur at position $x + r$

Checking Which Position Next ?

- So, when T[x..] gets a first mismatch after matching k chars with P, so that

  $$P[0..k-1] = T[x..x+k-1]$$

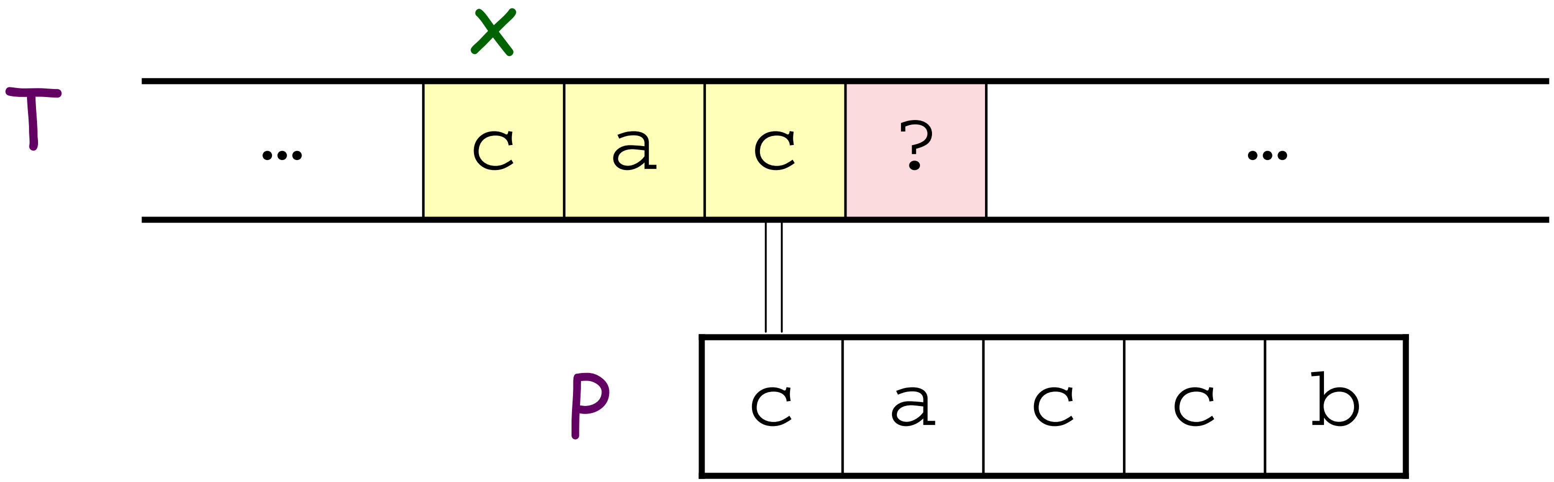
  we can restart the next checking at the leftmost position x+r such that

  $$T[x+r..x+k-1] \text{ is a prefix of } P$$

- Note: Leftmost x+r $\rightarrow$ smallest r

Key Observation

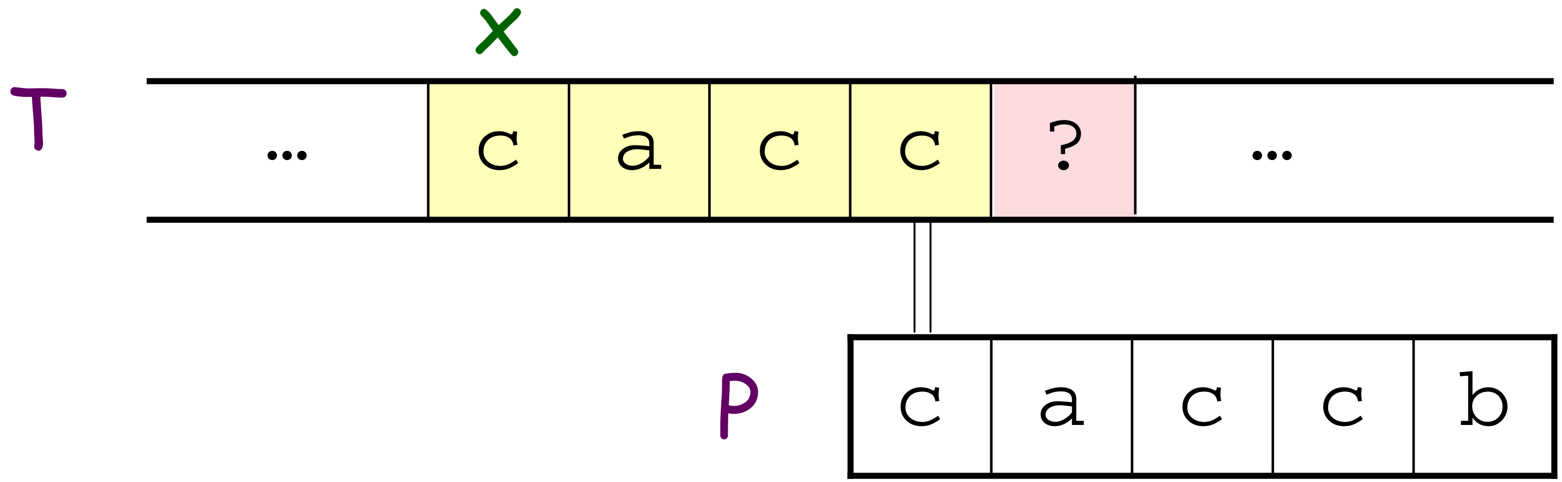
E.g., in our first example,



next checking can restart at pos x+2

Key Observation

In our second example,



next checking can restart at pos x+3

Finding Desired r

- We observe that

$$T[x+r..x+k-1] = P[r..k-1]$$

- So to find the desired r, we need the smallest r such that

$$P[r..k-1] \text{ is a prefix of } P$$

- What does that mean ??

Finding Desired r (Example 1)

P

c	a	c	c	b

When k = 3, we ask:

prefix of P ?
No …

	a	c		

prefix of P ?
Yes ! (r=2)

		c		

Finding Desired r (Example 2)

P

c	c	a	c	c

When k = 5 (what does that mean??), we ask:

prefix of P ?
No …

	c	a	c	c

prefix of P ?
No …

		a	c	c

prefix of P ?
Yes ! (r=3)

			c	c

Finding Desired r

- For each k, the smallest r such that

  P[r..k-1] is a prefix of P

  implies

  P[r..k-1] is longest such prefix

- Let us define a function π, called **prefix function**, such that

  π(k) = length of such P[r..k-1]

KMP Algorithm

- The KMP algorithm relies on the prefix function to locate all occurrences of P in $O(n)$ time → optimal !
- Next, we assume that the prefix function is already computed
  - We first describe a simplified version and then the actual KMP
- Finally, we show how to get prefix function

Simplified Version

Set $x = 0$;

while ($x < n-p+1$) {

1. Match T with P at position x ;
2. Let k = #matched chars ;
3. if ($k == p$) output "match at x" ;
4. Update $x = x + k - \pi(k)$;

}

What is the worst-case running time ?

How can we improve ?

- In simplified version, inside the while loop,

  Line 1 restarts matching (every char of)
  T with P from position x

- In fact, if previous step of while loop has matched k chars, we know in this round,

  the first π(k) chars are already matched

- What if we take advantage of this ??

KMP Algorithm

Set $x = 0$; $k = 0$;

while ($x < n-p+1$) {

1. Match T with P at position x
   but starting from $k+1^{th}$ position;
2. Update k = #matched chars;
3. if ($k == p$) output "match at x" ;
4. Update $x = x + k - \pi(k)$;
5. Update $k = \pi(k)$;

}

k keeps track of #matched chars

Running Time

- The running time comes from four parts:

1. Mis/matching a char of T with P (Line 1)
2. Updating the position x (Line 4)
3. Output match (Line 3)
4. Updating k (Line 2, Line 5)

Since each char is matched once, and x increases for each mismatch

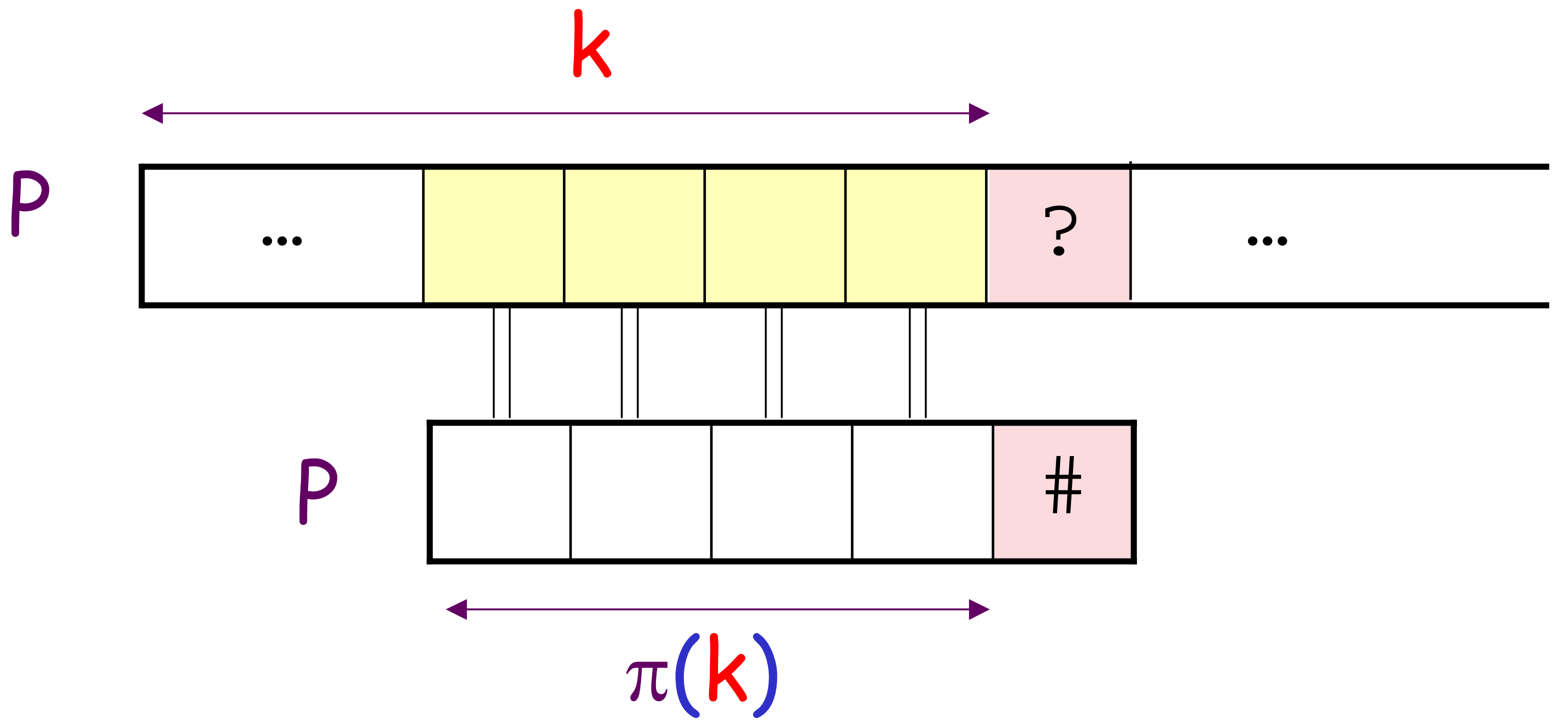
➔ in total $O(n)$ time

Computing Prefix Function

- It remains to compute the prefix function
- In fact, it can be computed incrementally
  (finding $\pi(1)$, then $\pi(2)$, then $\pi(3)$, and so on)
- For instance, suppose we have obtained
  $\pi(1)$, $\pi(2)$, … , $\pi(k)$ already
  ➔ How can we get $\pi(k+1)$?

Key Observation

We know that a prefix of length $\pi(k)$ — $P[0..\ \pi(k)-1\ ]$ — is the longest prefix matching the suffix of $P[0..k-1]$

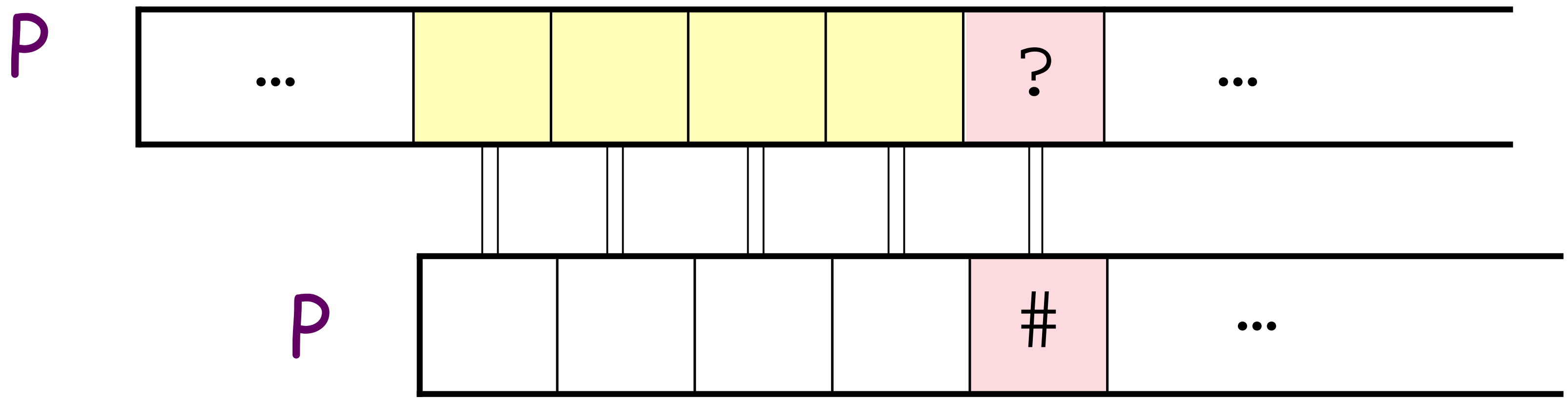


Key Observation

What if the next corresponding chars,

$P[\pi(k)]$ and $P[k]$

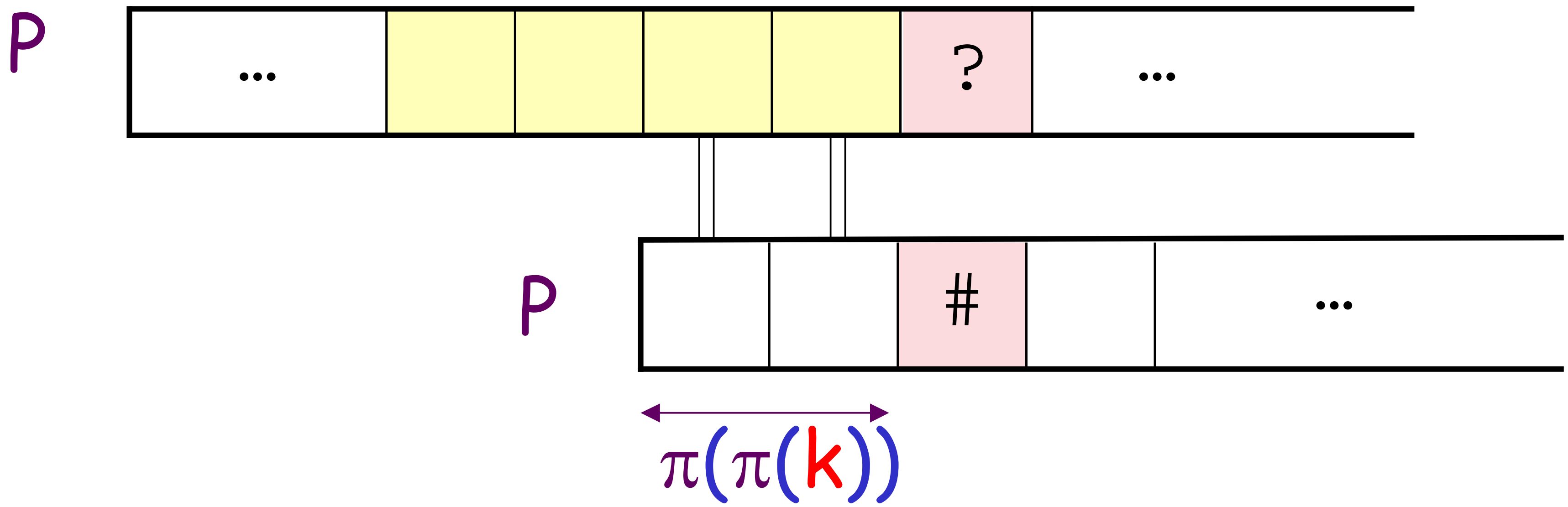
are the same ??



If same, $\pi(k+1) = \pi(k) + 1$ (prove by contradiction)

Key Observation

However, if P[π(k)] and P[k] are different, we should move the P below rightwards to search for the next longest prefix of P matching the suffix of P[0..k-1]

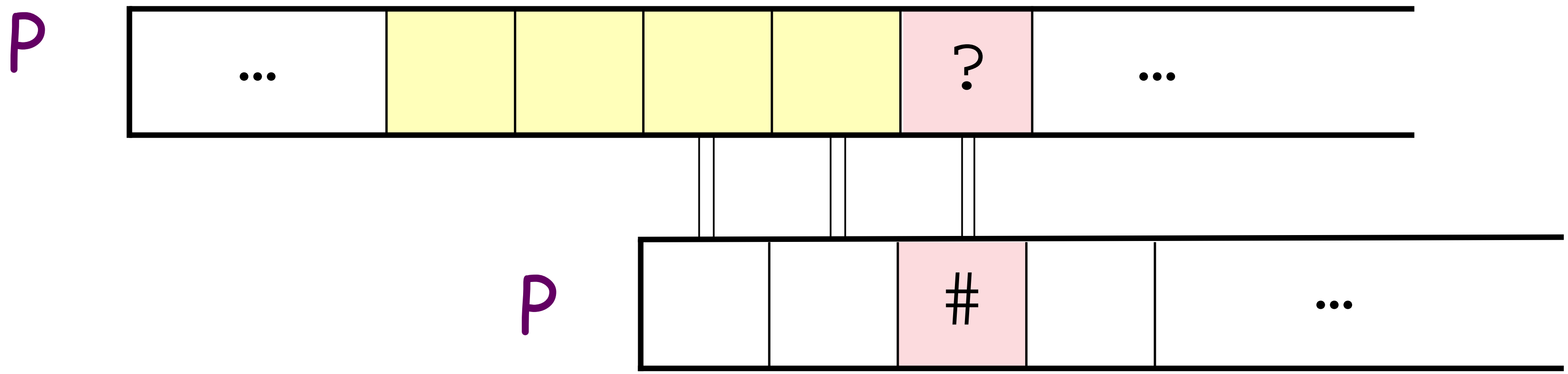


Key Observation

What if the next corresponding chars,

$P[\pi(\pi(k))]$ and $P[k]$

are the same ??



If same, $\pi(k+1) = \pi(\pi(k)) + 1$ (prove by contradiction)

Key Observation

- However, if $P[\pi(\pi(k))]$ and $P[k]$ are different, we see that we can repeat the procedure and obtain $\pi(k+1)$ when we find:

the longest prefix of P matching the suffix of $P[0..k-1]$, with its next char = $P[k]$

- Exactly the same as in string matching
- Total time : O(p) time

since (1) at most P matches, and

(2) P below moves rightwards for each mismatch