On the Complexity of Derivation in Propositional Calculus

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1. Formulation of the Problem and Principal Results

The question of the minimum complexity of derivation of a given formula in classical propositional calculus is considered in this article and it is proved that estimates of complexity may vary considerably among the various forms of propositional calculus. The forms of propositional calculus used in the present article are somewhat unusual, † but the results obtained for them can, in principle, be extended to the usual forms of propositional calculus.

The simplest objects in the calculi being considered are propositional variables which we combine into pairs (each variable occurs in one pair only; the number of such pairs is unlimited). The variables forming one pair will be called *conjugate* variables; if ξ denotes a variable, then the variable conjugate to it will be denoted by $\bar{\xi}$. Any finite set of variables will be called a *disjunction*; to represent a disjunction, we will write down all the variables contained in it on one line (in any order, without intervening signs); an empty disjunction will be denoted by Λ. Finally, we will be considering finite systems of disjunctions.

Let us assign to each variable one of two values: 1 (true) or 0(false); the assignment must be such that opposite values are assigned to conjugate variables. The value 1 is assigned to a disjunction if this

† An analogous system for predicate calculus has been introduced in [3].

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value has been assigned to at least one of the variables contained in it, otherwise the value 0 is assigned to it; in particular, the empty disjunction is always assigned the value 0. A system of disjunctions is said to be *satisfiable* if values can be assigned to variables in such a manner that all disjunctions of this system have the value 1; if a system of disjunctions is not satisfiable, we say that it is *contradictory*.

The calculi we shall use are aimed at establishing that systems of disjunctions are contradictory. The concept of a contradictory system of disjunctions is here an analog of the concept of an identically true formula in the usual propositional calculus. The problem of determining whether a given formula of propositional calculus is identically true can be reduced to the problem of determining whether a system of disjunctions is contradictory by reducing the negation of the formula to a conjunctive normal form. A transformation of this type, however, may lead to a considerable increase in the length of the formula, so that we will consider another method for transforming a formula of the propositional calculus into a system of disjunctions. Each subformula of a given formula will be associated with its own variable; two subformulas will be associated with conjugate variables if and only if one of the subformulas is a negation of the other. If a subformula $\mathcal{A}$ is the conjunction of subformulas $\mathfrak{B}$ and $\mathfrak{C}$ and these subformulas are associated with variables α, β, and γ, respectively, then the system of disjunctions $\bar{\alpha}\beta$, $\bar{\alpha}\gamma$, $\alpha\bar{\beta}\bar{\gamma}$ will be assigned to subformula $\mathcal{A}$. In an analogous manner we will assign systems of disjunctions to subformulas that are disjunctions and implications ($\alpha\bar{\beta}$, $\alpha\bar{\gamma}$, and $\bar{\alpha}\beta\gamma$ in the case of disjunctions and $\alpha\beta$, $\alpha\bar{\gamma}$, $\bar{\alpha}\bar{\beta}\gamma$ in the case of implications). Let us combine all of the systems of disjunctions obtained in this way and add the disjunction $\bar{\xi}$, where ξ is the variable associated with the whole of the given formula. It can be easily seen that the system of disjunctions will be contradictory if and only if the given formula is identically true.

Let a system of disjunctions be given. We will add new disjunctions to the system in accordance with definite rules in such a manner that the satisfiability of the supplemented system would follow from the satisfiability of the original system. This operation can be repeated many times. If we finally obtain the empty disjunction, then this will mean that the original system of disjunctions is contradictory; indeed, a system of disjunctions containing the empty disjunction cannot be satisfiable. We will consider two rules for extending a system of disjunctions without violating its satisfiability.

Rule 1 ("Annihilation"). If the system contains disjunctions $A\xi$ and $B\bar{\xi}$ (here ξ is a variable, A and B are disjunctions), then we can add the disjunction $A \cup B$ (here $\cup$ is used as the sign of set-theoretic union) to the system. Indeed, if the disjunctions $A\xi$ and $B\bar{\xi}$ have both received the value 1 for some selection of variable values, then the disjunction $A \cup B$ will also receive the value 1.

Rule 2 ("Extension"). If α, β, and γ are any variables and neither α nor $\bar{\alpha}$ occurs in any disjunction of the system, then the system can be supplemented by the following list of disjunctions: $\alpha\beta$, $\alpha\gamma$, $\bar{\alpha}\bar{\beta}\bar{\gamma}$. Indeed, if the values assigned to the variables are such that all disjunctions of the original system have the value 1, then this is independent of the value of the variable α; let us assign to α the value of the disjunction $\bar{\beta}\bar{\gamma}$ and then the newly introduced disjunctions will receive the value 1, whereas the values of the previous disjunctions will not be changed.

In order to apply the annihilation rule we need only know that two definite disjunctions are contained in the system. Therefore, if we use only this rule, then we can consider that we are dealing with a calculus in which the derivable objects are the individual disjunctions: The disjunctions contained in the original system play the part of axioms, the annihilation rule serves as a derivation rule, and the "aim" of the derivation is to obtain the empty disjunction. On the other hand, if we also make use of the extension rule for determining whether a system of disjunctions is contradictory, then we will consider that all applications of this rule precede all applications of the annihilation rule and the disjunctions obtained according to the extension rule will be equated to the axioms; thus, we consider that the extension rule is not a derivation rule, but a schema for the generation of new axioms.

Thus, we will consider the following calculus (which will be denoted by the letter $\mathfrak{A}$): The objects to be derived are disjunctions, the number of axioms is not fixed, and the only derivation rule allows us to derive the disjunction $A \cup B$ from the disjunctions $A\xi$ and $B\bar{\xi}$. An application of this rule will be called the *annihilation* of the variables ξ and $\bar{\xi}$. If we are given a system of disjunctions Σ, taking all of the disjunctions of Σ as axioms (no applications of the extension rule are allowed), and we can derive disjunction A in calculus

$\mathfrak{A}$, then we will write $\Sigma \vdash A$. The following theorem * concerning the completeness of $\mathfrak{A}$ can be easily proved.

Theorem 1. The system of disjunctions Σ is contradictory if and only if $\Sigma \vdash \Lambda$.

(The proof is by induction on the number of variables occurring in Σ; see Lemma 2 in Section 2.)

It can be seen from this theorem that the extension rule is not needed if we are merely interested in establishing that a given system of disjunctions is contradictory. However, as we shall see below, the use of this rule leads to a significant decrease in the complexity of derivations.

A derivation in the calculus $\mathfrak{A}$ (in the following we will simply say "derivation") can be represented either as a linear sequence of formulas or in the form of a tree. If the complexity of a derivation is measured by the number of occurrences of disjunctions in the derivation, then the linear and tree notations may lead to different values of the complexity because a given disjunction may appear more than once in a tree. We will always use the tree notation, but we will estimate complexity in two ways. The number of occurrences of axioms in a tree will be called L-*complexity* (since each occurrence of a disjunction, excluding axioms, has exactly two premises, the total number of occurrences of disjunctions in a tree is equal to two times the L-complexity minus 1). The number of distinct disjunctions occurring in a derivation will be called the N-*complexity* of this derivation (this obviously corresponds to the length of the derivation in linear notation).

If Σ is a system of disjunctions and A a disjunction such that $\Sigma \vdash A$, then $L_A(\Sigma)$ [$N_A(\Sigma)$] will denote the minimal L-complexity (N-complexity) of the derivation of disjunction A from Σ. We will be primarily interested in the case $A = \Lambda$ and then we will write $L(\Sigma)$ and $N(\Sigma)$.

Theorem 2. For an arbitrary contradictory system of disjunctions Σ, we have

$$\frac{N(\Sigma)+1}{2} \leq L(\Sigma) \leq \left(\frac{3}{2}\right)^{N(\Sigma)-1}$$

* See also [3].

Here, the upper bound on $L(\Sigma)$ is obtained as follows. Linear notation is used to write down the derivation of Λ from Σ with minimum N-complexity and induction on n is used to prove that if A is the n-th disjunction in this derivation, then $L_A(\Sigma) \leq [(\frac{3}{2})^{n-1}]$; some rearrangement is possible within the given derivation.

Each occurrence of a disjunction in a derivation corresponds to a subtree - the part of the complete tree forming the derivation of the given occurrence. We will say that a derivation is *regular* if each of its subtrees satisfies the following condition: The subtree does not contain any annihilations of variables occurring in the final disjunctions of the subtree. In the case of the derivation of the empty disjunction, the regularity condition is equivalent to the condition that an annihilation of the same variable does not occur twice on one branch of the derivation.

If Σ is a system of disjunctions for which $\Sigma \vdash \Lambda$, then $\bar{L}(\Sigma)[\bar{N}(\Sigma)]$ will denote the minimum L-complexity (N-complexity) of a regular derivation of Λ from Σ. It is obvious that we have

$\bar{L}(\Sigma) \geq L(\Sigma)$ and $\bar{N}(\Sigma) \geq N(\Sigma)$. Moreover, it is obvious that we have $\bar{L}(\Sigma) \geq \frac{N(\Sigma)+1}{2}$.

The following theorem also holds.

<u>*Theorem 3.*</u> *For an arbitrary contradictory system of disjunctions Σ, we have*

$$\bar{L}(\Sigma) = L(\Sigma) .$$

In other words, L-complexity is always a minimum for regular derivations. The proof of this is based on the following assertion: If $\Sigma \vdash A$, then from any disjunction B we can construct a disjunction C such that $C \subset (A \cup B)$ and there exists a regular derivation of C from Σ with an L-complexity not greater than $L_A(\Sigma)$ [proved by induction on $L_A(\Sigma)$].

Whether there exists a contradictory system of disjunctions Σ such that $\bar{N}(\Sigma) > N(\Sigma)$ remains an open question.

Finally, let us define the quantities $L^*(\Sigma)$, $N^*(\Sigma)$ and $\bar{N}^*(\Sigma)$ for an arbitrary contradictory system of disjunctions as the minimum values of the quantities $L(\Sigma^*)$, $N(\Sigma^*)$ and $\bar{N}(\Sigma^*)$, respectively, where Σ^* stands for any systems of disjunctions obtained from Σ by the (repeated) application of the extension rule [to define these quantities, it is obviously sufficient to consider only those systems Σ^* for the construction of which not more than $L(\Sigma)$ new variables have been introduced]. It is obvious that $L^*(\Sigma) \leq L(\Sigma)$, etc. Finally we will denote the number of disjunctions in system Σ by $\lambda(\Sigma)$.

The principal results of the present article refer to the connections between the various quantities.

<u>*Theorem 4.*</u> *For any number c less than $\frac{1}{4}$, and arbitrary n, we can find a contradictory system of disjunctions Σ for which*

$$\log_2 L(\Sigma) \geq c(\log_2 \bar{N}(\Sigma))^2 > n ,$$

$$\log_2 L(\Sigma) \geq c(\log_2 L^*(\Sigma))^2 > n .$$

<u>*Theorem 5.*</u> *There exists a positive number c such that for any n we can find a contradixtory system of disjunctions Σ for which*

$$\log_2 \bar{N}(\Sigma) \geq c \sqrt[3]{\bar{N}^*(\Sigma)} > n ,$$

$$\log_2 \bar{N}(\Sigma) \geq c \sqrt{\lambda(\Sigma)} > n .$$

It follows from these theorems that, in particular, $L(\Sigma)$ is not majorized by any power function of $\bar{N}(\Sigma)$ and $L^*(\Sigma)$, while $\bar{N}(\Sigma)$, in its turn, is not majorized by any power function of $\bar{N}^*(\Sigma)$ and $\lambda(\Sigma)$ [and all the more so by $L(\Sigma)$]. It would be of great interest to establish the analogous relation between $\bar{N}^*(\Sigma)$ and $\lambda(\Sigma)$, but we were unable to do this by the methods used in the present article. The method by which systems of disjunctions satisfying the conditions of Theorems 4 and 5 can be constructed will be described in the following sections.

We can establish some connections between the quantities being considered here and the properties of the derivations in the sequential form of the propositional calculus with the help of the following construction. Suppose we are given the derivation of a formula in sequential calculus. With each formula that is a member of some sequent of this derivation we associate a variable (conjugate variables are only associated with formulas that are negations of one another). Replacing all formulas in each sequent by the corresponding variables and transferring all terms into the succedent (obviously, replacing all variables by conjugate variables), we obtain a tree consisting of disjunctions. If axioms are introduced in the same way as was done in the case of the transformation of a conventional formula into a system of disjunctions, then the tree we obtain can be easily transformed into the derivation of the empty disjunction from these axioms in the calculus $\mathfrak{A}$. If the original derivation was cut free, then the collection of axioms for derivation in $\mathfrak{A}$ can only be constructed from the subformulas of the final formula of the original derivation; this reveals the analogy between the use of the cut rule and the use of the rule for the extension of the axiom system preceding a derivation in $\mathfrak{A}$. The analog of the concept of a regular derivation in $\mathfrak{A}$ is the concept of a "weeded-out" derivation in sequential calculus introduced in [2] (this concept is formulated in terms of restrictions on the repetition of side formulas on a single branch of the derivation).

A heuristic interpretation can also be given to various restrictions on a derivation described in terms of the calculus $\mathfrak{A}$. Thus, the introduction of L-complexity instead of N-complexity obviously corresponds to a prohibition on the multiple use in a proof of a result once it is obtained (to use it a second time, we must repeat all of the proof of the intermediate result). The use of the extension rule corresponds to the admission of auxiliary constructions in the proof. Finally, the regularity condition can be interpreted as a requirement for not proving intermediate results in a form stronger than that in which they are later used (if A and B are disjunctions such that $A \subseteq B$, then A may be considered to be the stronger assertion of the two; if the derivation of a disjunction containing a variable ξ involves the annihilation of the latter, then we can avoid this annihilation, some of the disjunctions in the derivation being replaced by "weaker" disjunctions containing ξ. * The theorems proved above throw some light on the influence of the above restrictions on the complexity of a proof.

2. *Systems of Disjunctions Connected with Graphs*

By a *graph* ** we here mean a finite symmetric (i.e. unoriented) graph without loops (not necessarily connected). We will assume, in addition, that each edge of the graph is associated with a pair of conjugate variables (distinct pairs being associated with distinct edges).

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* In addition to the appearance of ξ, the disappearance of other variables is possible in the course of such a rearrangement.

** The terminilogy of [1] is used in this article.

A *labeled graph* is a graph supplied with the following additional information: Each vertex is assigned one of two logical values (0 or 1) and for each edge we select one of the corresponding pair of conjugate variables. If Φ is a labeled graph, then the original (unlabeled) graph will be denoted by $|\Phi|$.

Suppose that the variables $\xi_1,\ldots,\xi_k$ (wit $k \geq 0$) are all distinct and among them there are no conjugate-variable pairs; let ε denote either 0 or 1. We will use $[\xi_1\ldots\xi_k]^\varepsilon$ to denote a system of disjunctions constructed from the variables $\xi_1,\ldots,\xi_k$ and variables conjugate to them, the disjunctions of this system satisfying the following conditions: 1) for each i (where $1 \leq i \leq k$), the disjunction contains either ξ_i or $\bar{\xi}_i$, but not both of these variables, 2) the number of values of i which correspond to occurrences of $\bar{\xi}_i$ in the disjunction is of opposite parity to the number ε. When $k > 0$, the system $[\xi_1\ldots\xi_k]^\varepsilon$ obviously consists of 2^{k-1} disjunctions. Further, if the logical values $\varepsilon_1,\ldots,\varepsilon_k$ are assigned to the variables $\xi_1,\ldots,\xi_k$, then for all disjunctions of $[\xi_1\ldots\xi_k]^\varepsilon$ to be true it is necessary and sufficient that

$$\varepsilon_1 \oplus \ldots \oplus \varepsilon_k = \varepsilon$$

(here, the sign $\oplus$ denotes addition modulo 2).

Let Φ be a labeled graph. Let us construct for each vertex of Φ a system of disjunctions $[\xi_1\ldots\xi_k]^\varepsilon$, where ε is the logical value assigned to the given vertex and $\xi_1,\ldots,\xi_k$ are the variables that have been selected for the edges that are incident to the given vertex. The union of all such systems of disjunctions for all vertices of Φ will be denoted by $\alpha(\Phi)$. We will denote the sum modulo 2 of the logical values assigned to all vertices of Φ by $\sigma(\Phi)$.

Let us establish the extent to which the properties of the system of disjunctions $\alpha(\Phi)$ depend on the graph $|\Phi|$. Let us first of all note that if in Φ we change the labeling of one edge (i.e. if we select for it a variable conjugate to the original variable) and simultaneously change the logical values at the ends of this edge, then $\alpha(\Phi)$ will remain unchanged. Analogously, if two vertices of Φ are connected by a (simple) chain, then, without changing $\alpha(\Phi)$, we can interchange the logical values at both of these vertices and simultaneously change the labeling of all edges forming this chain. It follows from this that if $|\Phi|$ is a connected graph, then any change of logical values at the vertices Φ preserving $\sigma(\Phi)$ can be compensated by a suitable change of the edge labeling so that $\alpha(\Phi)$ does not change. We see from this that if the graph $|\Phi|$ is connected and $\sigma(\Phi) = 0$, then the system of disjunctions $\alpha(\Phi)$ is satisfiable. Indeed, in this case we can consider that all vertices have been assigned the logical value 0 and to make all disjunctions belonging to $\alpha(\Phi)$ true, it is sufficient to assign the value 0 to all the variables selected for the edges. On the other hand, if $\sigma(\Phi) = 1$, then the system of disjunctions $\alpha(\Phi)$ is contradictory. Indeed, if we could assign values to variables in such a manner that all disjunctions belonging to $\alpha(\Phi)$ were true, then for each vertex the sum modulo 2 of the values of the variables assigned to the edges incident to this vertex would be equal to the logical value assigned to the given vertex; if we sum these equalities modulo 2 for all variables, we will obtain 0 for the left-hand side, since the value of the variable for each edge enters the sum exactly twice, whereas the right-hand side will be equal to $\sigma(\Phi)$, i.e. 1.

Thus, if we are given a connected graph Γ, then a contradictory system of disjunctions $\alpha(\Phi)$ corresponds to every labeled graph Φ such that $|\Phi| = \Gamma$ and $\sigma(\Phi) = 1$. Moreover, the systems of disjunctions corresponding to various graphs of this type can be interconverted by a renaming of variables, so that the properties of the system of disjunctions $\alpha(\Phi)$ are in essence completely determined by the graph Γ. The values of the functions L, N, $\bar{N}$, L^*, N^*, $\bar{N}^*$ and λ introduced in Section 1 will depend only on Γ for the system of disjunctions $\alpha(\Phi)$, where $|\Phi| = \Gamma$ and $\sigma(\Phi) = 1$, in view of which we will denote them by $L(\Gamma)$, $N(\Gamma)$, $\bar{N}(\Gamma)$, $L^*(\Gamma)$, $N^*(\Gamma)$, $\bar{N}^*(\Gamma)$ and $\lambda(\Gamma)$, respectively.

$\lambda(\Gamma)$ can be easily calculated directly. An upper bound to $\bar{N}^*(\Gamma)$ can be obtained on the basis of the fact that the discussion given above for proving that the system of disjunctions $\alpha(\Phi)$ is contradictory by means of modulo 2 addition can be formalized in the calculus $\mathfrak{A}$ by means of the rule for extending the axiom system; this yields for $\bar{N}^*(\Gamma)$ an upper bound in the form of a linear function of $\lambda(\Gamma)$ and the product of the number of vertices of Γ times the number of its edges. The upper bound to $L^*(\Gamma)$ obtained in the same way is found to be considerably worse: in the general case it contains the number of edges as an exponent of 2. It can also be noted that each of the above quantities is not less than the number of vertices of Γ, since in any derivation of Λ from $\alpha(\Phi)$ at least one axiom is used for each vertex of Φ.

In the case of $L(\Gamma)$ and $\bar{N}(\Gamma)$, it is possible to write exact recurrence formulas for them.

We will use the letter a as a variable for edges of a graph; letters Γ and Δ (possibly subscripted) will be used as variables for connected graphs. If a is an edge of graph Γ (we write $a \in \Gamma$), then the partial graph obtained by the removal of this edge from Γ (the number of vertices being unchanged) either remains a connected graph and we will then denote it by Γ_a, or it splits into two connected components which we will denote (in arbitrary order) by Γ'_a and Γ''_a (in this case we will say that the edge a *ruptures* Γ).

<u>*Theorem 6.*</u> *The following recurrence relation holds:*

$$\mathsf{L}(\Gamma) = \begin{cases} 1 \text{ if } \Gamma \text{ consists of one vertex only,} \\ \min\limits_{a \in \Gamma} \mathsf{L}^a(\Gamma) \text{ otherwise,} \end{cases}$$

where

$$\mathsf{L}^a(\Gamma) = \begin{cases} 2\mathsf{L}(\Gamma_a), & \text{if } a \text{ ruptures } \Gamma, \\ \mathsf{L}(\Gamma'_a) + \mathsf{L}(\Gamma''_a), & \text{if } a \text{ does not rupture } \Gamma. \end{cases}$$

Let us introduce the following terminology to assist us in obtaining a recurrence relation for $\bar{\mathsf{N}}(\Gamma)$. The *cyclomatic number* of graph Ξ [denoted by $\nu(\Xi)$] is the number of edges of the graph minus the number of vertices plus the number of connected components (i.e. the cyclomatic number is the rank of the one-dimensional group of cycles). Let Δ be a partial subgroup of graph Γ. We will say that edge a of Γ *emerges from* Δ if a does not belong to Δ, but has at least one vertex in common with Δ. Let $\Gamma \setminus \Delta$ denote the subgraph of Γ obtained by the removal of all vertices of Δ (together with all edges incident to these vertices). Let us take

$$S_\Gamma(\Delta) = \nu(\Gamma) - \nu(\Delta) - \nu(\Gamma \setminus \Delta).$$

It can be easily seen that $S_\Gamma(\Delta)$ is the maximum number of edges emerging from Δ that can be removed without disrupting the connectedness of Γ. Let us define a function Z_Γ on the partial subgraphs of graph Γ as follows:

$$Z_\Gamma(\Delta) = \begin{cases} 2^{S_\Gamma(\Delta)}, & \text{if } \Delta \text{ consists of one vertex only,} \\ 2^{S_\Gamma(\Delta)} + \min\limits_{a\in\Delta} Z_\Gamma^a(\Delta) & \text{otherwise,} \end{cases}$$

where

$$Z_\Gamma^a(\Delta) = \begin{cases} Z_\Gamma(\Delta_a), & \text{if } a \text{ does not rupture } \Delta, \\ Z_\Gamma(\Delta'_a) + Z_\Gamma(\Delta''_a), & \text{if } a \text{ ruptures } \Delta. \end{cases}$$

Theorem 7. *For any connected graph* Γ, *we have*

$$\bar{N}(\Gamma) = Z_\Gamma(\Gamma).$$

Let us outline the main stages of the proofs of Theorems 6 and 7. In the following, we will only be considering disjunctions which do not simultaneously contain two conjugate variables; the letters A, B, and C will be used as variables for such disjunctions. The letter Σ will be used as a variable for systems of disjunctions, the letters Φ, Ψ will be used for connected labeled graphs, and the letters ξ, η for propositional variables.

We will use $\Sigma \setminus \xi$ to denote the system of disjunctions obtained from Σ as follows: The disjunctions belonging to Σ and containing $\bar{\xi}$ are removed, while ξ (if it occurs) is deleted from the other disjunctions.

Lemma 1. *If we are given a regular derivation of a one-element disjunction* ξ *from the system of disjunctions* Σ, *then by deleting all occurrences of* ξ *in this derivation we obtain a regular derivation of* Λ *from* $\Sigma \setminus \xi$.

Lemma 2. *If we have* $\Sigma \setminus \xi \vdash \Lambda$, *then one of the following cases holds:*

$$1)\ \Sigma \vdash \xi \quad \text{and} \quad L_\xi(\Sigma) \le L(\Sigma \setminus \xi)\ ,$$

$$2)\ \Sigma \vdash \Lambda \quad \text{and} \quad L(\Sigma) \le L(\Sigma \setminus \xi)\ .$$

We will use $\Phi \setminus \xi$ to denote the labeled graph obtained from Φ as follows: 1) If the variable ξ has been selected for one edge of Φ, then this edge is removed from Φ (all vertices being left intact); 2) if the variable $\bar{\xi}$ has been selected for one of the edges of Φ, then this edge is removed from Φ and simultaneously the logical values assigned to the ends of this edge are inverted; 3) in all other cases $\Phi \setminus \xi = \Phi$. It should be noted that for all ξ, η such that $\eta \neq \bar{\xi}$, and for arbitrary Φ, we have

$$(\Phi \setminus \xi) \setminus \eta = (\Phi \setminus \eta) \setminus \xi \ .$$

In view of this, we can define $\Phi \setminus A$ for any Φ and A by induction on the number of variables in A; thus

$$\Phi \setminus \Lambda = \Phi \ ,$$

$$\Phi \setminus A\xi = (\Phi \setminus A) \setminus \xi \ .$$

We define a system of disjunctions $\Sigma \setminus A$ for any Σ and A in an analogous manner.

Lemma 3. $\sigma(\Phi \setminus A) = \sigma(\Phi)$.

Lemma 4. $\alpha(\Phi \setminus A) = \alpha(\Phi) \setminus A$.

Lemma 5. If Ξ is an unconnected labeled graph and we are given a derivation of a disjunction from $\alpha(\Xi)$ then we can identify a connected component Ψ such that the given derivation is a derivation from $\alpha(\Psi)$.

The proof of Theorem 6 is based on Lemma 1 - 5, Lemmas 3 and 4 being only used for one-element disjunctions.

In the following, Φ will be taken to be a fixed labeled graph such that $|\Phi| = \Gamma$ and $\sigma(\Phi) = 1$. If for some edge of Φ we select a variable ξ or $\bar{\xi}$, then this edge will be denoted by $|\xi|$. We will say that a disjunction A is *admissible* (for Φ) if the labeled graph $\Phi \setminus A$ contains exactly one connected component Ψ for which $\sigma(\Psi) = 1$. This component will be denoted by Φ_A.

Lemma 6. If a disjunction A is admissible and $|\xi| \in |\Phi_A|$, then the disjunction $A\xi$ is also admissible and $\Phi_{A\xi}$ is a connected component of $\Phi_A \setminus \xi$.

Let $\mathcal{D}$ be a fixed regular derivation of Λ from $\alpha(\Phi)$. The *index of occurrence* of subtree $\mathcal{D}'$ in the tree $\mathcal{D}$ is the disjunction defined as follows: 1) The index of occurrence of $\mathcal{D}$ in $\mathcal{D}$ is Λ; 2) if A is the index of occurrence of $\mathcal{D}'$ in $\mathcal{D}$, the premises of the last step in $\mathcal{D}'$ are of the form $B\xi$ and $C\bar{\xi}$, and their subtrees are $\mathcal{D}_1'$ and $\mathcal{D}_2'$, then the index of the corresponding occurrence of $\mathcal{D}'_1$ will be $A\xi$ (and the index of occurrence of $\mathcal{D}'_2$ will be $A\bar{\xi}$). The term *index of occurrence of a disjunction* in $\mathcal{D}$ will be used for the index of occurrence of the corresponding subtree.

Lemma 7. *If A is the index of occurrence of subtree $\mathcal{D}'$ in $\mathcal{D}$, then 1) disjunction A is admissible,*
2) if we delete from $\mathcal{D}'$ all occurrences of variables appearing in the conclusion of $\mathcal{D}'$, we obtain a regular derivation of Λ from $\alpha(\Phi_A)$.

This lemma is proved by induction on A with the use of Lemmas 1 and 4 - 6.

Lemma 8. *If A is the index of occurrence in $\mathcal{D}$ of a subtree consisting of a single axiom, then Φ_A consists of only one vertex.*

This is a consequence of Lemma 7.

Let A be an admissible disjunction. Let $\delta(A)$ denote the set of variables ξ such that $\xi \in A$ and the edge $|\xi|$ has a vertex in common with Φ_A [$\delta(A)$ contains as many variables as there are edges emerging from $|\Phi_A|$ considered as a partial subgraph of Γ]. The following lemma shows that this disjunction can be found from the index of occurrence of the disjunction in $\mathcal{D}$ without any knowledge of $\mathcal{D}$ itself.

Lemma 9. *If A is the index of occurrence of a disjunction B in $\mathcal{D}$, then $B = \delta(A)$.*

The proof of this lemma is based on Lemma 7.

Lemma 10. *If A is an admissible disjunction and $\delta(A) \subseteq B \subseteq A$, then B is also an admissible disjunction, $\Phi_B = \Phi_A$ and $\delta(B) = \delta(A)$.*

Lemma 11. *Let A be the index of occurrence of some subtree $\mathcal{D}'$ in $\mathcal{D}$. Then, for any distance B we have*

$$\sum_C 2^{\nu(|\Phi \setminus C|)} \leq 2^{\nu(|\Phi \setminus (A \cup B)|)} ,$$

where the summation extends over the indices C of all occurrences of B in $\mathcal{D}$ referring to subtree $\mathcal{D}'$.

This lemma is proved by induction on the L-complexity of $\mathcal{D}'$. Lemmas 9 and 10, as well as the following fact are used: When an edge is deleted from a graph, the cyclomatic number remains unchanged if the deleted edge ruptures one of the connected components, otherwise the cyclomatic number decreases by one (see [1], Theorem 1 of Chapter 4).

Lemma 12. *For any disjunction* B, *we have*

$$\sum_{C} 2^{\nu(|\Phi \smallsetminus C|)-\nu(|\Phi \smallsetminus B|)} \leq 1,$$

where the summation extends over the indices C *of all occurrences of disjunction* B *in* $\mathcal{D}$.

Lemma 13. *The* N*-complexity of derivation* $\mathcal{D}$ *is not less than*

$$\sum_{C} 2^{\nu(|\Phi \smallsetminus C|)-\nu(|\Phi \smallsetminus \delta(C)|)}$$

where the summation extends over the indices C *of all occurrences of disjunctions in* $\mathcal{D}$.

Lemma 14. *If* A *is the index of occurrence of a subtree* $\mathcal{D}'$ *in* $\mathcal{D}$, *then we have*

$$2^{\nu(\Gamma)} \cdot \sum_{C} 2^{\nu(|\Phi \smallsetminus C|)-\nu(|\Phi \smallsetminus \delta(C)|} \geq 2^{\nu(|\Phi \smallsetminus A|)} \cdot Z_{\Gamma}(|\Phi_A|) ,$$

where the summation extends over the indices C *of all occurrences in* $\mathcal{D}$ *of disjunctions referring to the subtree* $\mathcal{D}'$.

It can be easily proved with the help of Lemmas 13 and 14 that $\bar{N}(\Gamma) \geq Z_{\Gamma}(\Gamma)$. In order to prove the inverse inequality, let us select in each partial subgraph Δ of graph Γ with more that one vertex, an edge a which minimizes $Z_{\Gamma}^{a}(\Delta)$. Let us introduce the concept of a *regular partial subgraph* of graph Γ as follows: Γ is regular; if Δ is regular and an edge a has been selected in Δ, then Δ_a is regular (if a does not rupture Δ) or Δ_a' and Δ_a'' are regular (if a ruptures Δ). We will say that a disjunction A *belongs to* a partial subgraph Δ of graph Γ if A is admissible disjunction, $\delta(A) = A$ and $|\Phi_A| = \Delta$. The inequality $\bar{N}(\Gamma) \leq Z_{\Gamma}(\Gamma)$ then follows immediately from the following three assertions.

Lemma 15. *We can construct a regular derivation of Λ from $\alpha(\Phi)$ which contains only disjunctions belonging to the regular partial subgraphs of graph Γ.*

Lemma 16. *The number of distinct disjunctions belonging to the partial subgraph Δ of graph Γ is equal to*

$$2^{S_\Gamma(\Delta)} .$$

Lemma 17. *We have*

$$Z_\Gamma(\Gamma) = \sum_\Delta 2^{S_\Gamma(\Delta)} ,$$

where the summation extends over all regular partial subgraphs Δ of graph Γ.

3. Bounds for Concrete Graphs

We will consider graphs P_{kl} (where k and l are positive integers) defined as follows: the vertices of graph P_{kl} are points on a plane with integer coordinates x,y such that $1 \leq x \leq k$ and $1 \leq y \leq l$ (in all there are kl vertices) and the edges of P_{kl} are segments of unit length drawn parallel to the coordinate axes. We will usually assume that $k \geq l$.

Theorem 8. *There exist positive constants c_1 and c_2 such that for arbitrary integers k and l we have*

$$\left(\frac{c_1 k}{l}\right)^l < L(P_{kl}) < \left(\frac{c_2 k}{l}\right)^l .$$

Let us give the principal stages in the proof of this theorem.

Lemma 18. *Let Γ' be a connected partial subgraph of graph Γ and Δ a connected partial subgraph of Γ'. We then have*

$$S_\Gamma(\Delta) \geq S_{\Gamma'}(\Delta) .$$

Let us fix a value for l greater than one and also a sufficiently large m; we assume that $\Gamma = P_{ml}$. Let Δ be a connected partial subgraph of Γ. Let $w(\Delta)$ denote the number of distinct abscissas of vertices of Δ. We will call Δ an *internal graph* if Δ does not contain vertices with abscissas 1 and m.

Lemma 19. *Let Δ be an internal graph and let a be an edge that ruptures Δ. Then, we have*

$$S_\Gamma(\Delta)-S_\Gamma(\Delta'_a) \geq \begin{cases} l-1, & \text{if } \Delta''_a \text{ contains vertices with all integer ordinates from } 1 \text{ to } l \\ w(\Delta)-w(\Delta'_a) & \text{otherwise.} \end{cases}$$

The proof of this lemma is based on Lemma 18, as well as the relation

$$S_\Gamma(\Delta)-S_\Gamma(\Delta'_a) = S_{\Gamma\setminus\Delta'_a}(\Delta''_a) .$$

Lemma 20. *If the constant* c *is such that the inequality*

$$L(\Delta) \geq c\cdot 2^{-S_\Gamma(\Delta)}\cdot(w(\Delta))^{1}$$

is satisfied for all internal graphs Δ satisfying the condition $w(\Delta) < 2l$, *then it is satisfied for all internal graphs Δ.*

The proof of this lemma is by induction on the number of edges of Δ, together with the use of Theorem 6 and Lemma 19.

Having chosen a suitable value for c, we obtain the required lower bound to $L(P_{kl})$ (for $k \leq m-2$) with the help of Lemma 20. The upper bound to $L(P_{kl})$ is obtained directly from Theorem 6: We must successively remove edges from P_{kl} until the graph P_{kl} is found to be cut into approximately two equal parts by the axis of abscissas.

The following bounds are also obtained by direct counting:

$$\bar{N}(P_{kl}) \leq ckl\cdot 2^{l} \quad ,$$

$$L^*(P_{kl}) \leq ckl\cdot 2^{l} \quad ,$$

$$\bar{N}^*(P_{kl}) \leq ckl^{2} \quad ,$$

$$\lambda(P_{kl}) \leq ckl$$

(here c is a constant). We can make use of Theorem 7 to obtain the bound on $\bar{N}(P_{kl})$; in this case, we must remove "exterior" edges from P_{kl} in such a manner that vertices are "lost" one by one in order of increasing abscissas. In order to construct sufficiently simple derivations involving the use of the rule for the extension of the axiom system, we can make use of the following idea (in fact, the bounds given below were obtained with the help of a slightly different construction). Additional axioms are constructed in such a manner that for each integer m such that $1 \leq m < k$ we define a new variable ξ_m whose value is equal to the sum modulo 2 of the values of all variables selected for the edges that join vertices with abscissas m and m+1; we then, first of all, derive simple relations between ξ_m and ξ_{m+1} (the complexity of such a derivation depends only on l) and only then derive Λ from them.

To prove Theorem 4, it is now sufficient to consider the graph P_{kl} with $k = l \cdot 2^l$. Theorem 5 can be obtained if we set k = l and make use of the following theorem.

Theorem 9. For any positive integer k, *we have*

$$\bar{N}(P_{kk}) \geq 2^{k-1} .$$

To prove this theorem, let us set $\Gamma = P_{kk}$ and once more make use of the concept of a regular partial subgraph introduced to prove Theorem 7. In view of Theorem 7 and Lemma 17, our assertion will be proved if we can find a regular partial subgraph Δ of graph Γ such that $S_\Gamma(\Delta) \geq k-1$. To find such a Δ, we will construct regular partial subgraphs of Γ by successively deleting edges of Γ; each time that the deleted edge ruptures the partial subgraph being considered, we will select that component which has more vertices in common with the "perimeter" (either one, if the numbers of common vertices are equal). By this method we will construct a regular partial subgraph in which the number of vertices in common with the "perimeter" is greater than $\frac{4}{3}(k-1)$, but not greater than $\frac{8}{3}(k-1)$; this will be the required subgraph.

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